On even-H-free colorings

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Abstract

For a given graph H, an edge-coloring $\mathcal{C}$ of the complete graph K_n is *even-H-free* if every copy of H intersects at least one color class with an odd number of edges. These colorings are motivated by the study of graph codes, which were introduced in [AGK+22]. Even-H-free colorings exhibit interesting properties and are closely related to other problems in Ramsey Theory. The present short paper includes new results about the extremal properties of such colorings. Let $g(n, H)$ be the minimum number of colors required in an even-H-free coloring of K_n. Our main result is the identification of a class of graphs H, namely forest of even stars satisfying appropriate size constraints, for which $g(n, H) = \Theta(n^2)$.

1 Introduction

A binary Error Correcting Code is a family of binary strings $\mathcal{C} \subseteq \mathbb{F}_2^n$ where for each $x, y \in \mathcal{C}$, $\delta(x, y) \geq d$ where $\delta(x, y)$ is the Hamming distance between x and y, namely the number of positions i in which the bits $x[i]$ and $y[i]$ differ. Error correcting codes have applications in communication protocols, data storage, and theoretical computer science. These codes have tight connections with questions and results in combinatorics. In particular, the papers [AS05], [BE21], [BZ04], [DEL+22] contain breakthrough results in coding theory that are closely related to ideas and techniques in graph theory. This suggests further study of the interplay between graph theory and coding theory.

The authors of [AGK+22] introduced a new version of the classical problems of coding theory. They proposed a variant of the classical binary error correcting codes problems in which codewords are required to differ according to some prescribed structure rather than solely in a required minimum number of coordinates. Viewing binary strings as indicator vectors of edge sets of graphs, one can formulate the problem of structurally different binary code families in terms of graphs. This alternative versions of codes are

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termed graph-codes. More precisely, let $V = [n] = \{1, 2, \ldots, n\}$ and let $\mathcal{H}$ be a family of graphs on the set of vertices $[n]$ which is closed under isomorphism. A collection of graphs $\mathcal{F}$ on $[n]$ is called an $\mathcal{H}$-(graph)-code if it contains no two members whose symmetric difference is a graph in $\mathcal{H}$. The symmetric difference of two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices V is the graph $(V, E_1 \oplus E_2)$, where $E_1 \oplus E_2$ is the symmetric difference between E_1 and E_2, that is, the set of all edges that belong to exactly one of the two graphs. Note that this is equivalent to addition over $\mathbb{F}_2$ of the characteristic vectors of the sets of edges of the two graphs. In the special case that $\mathcal{H}$ contains all copies of a single graph H on $[n]$, this is called an H-code.

One noteworthy problem pertaining to graph codes is that of determining or estimating the maximum possible cardinality of different graph codes. Following the notation of [Alo23b], let $D_{\mathcal{H}}(n)$ denote the maximum possible cardinality of a $\mathcal{H}$-graph code on $[n]$ and put $d_{\mathcal{H}}(n) = \frac{D_{\mathcal{H}}(n)}{2^{\binom{n}{2}}}$. This is the maximum possible fraction of the total number of graphs on $[n]$ in an $\mathcal{H}$-code. Tight lower bounds for families $\mathcal{H}$ are many times obtained by *linear graph-codes*, which are families of graphs on $[n]$ closed under symmetric difference. In [Alo23b] it is shown that a linear graph code exhibits a rather large lower bound for the family $\mathcal{K}$ of all cliques, and in [AGK$^+$22] it is proved that a linear graph code exhibits a tight lower bound for the family of all disconnected graphs on the vertex set $[n]$. The prevalence of linear graph-codes motivates further study of their properties, indeed this is similar to the crucial role of linear codes in standard coding theory.

Linear graph codes are closely related to Ramsey-theoretic graph coloring problems, as shown in [Alo23b]. Specifically, for a given graph H and an edge coloring $\mathcal{C}$ of the complete graph, call a copy of H even if every color if every color appears in it an even number of times. An edge coloring is even-H-free if there is no even copy of H. It is not too difficult to show that there exists a linear H-code $\mathcal{F}$ with $|\mathcal{F}| \geq \frac{2^{\binom{n}{2}}}{f(n)}$ if and only if there exists a coloring of K_n with $f(n)^{c(H)}$ colors that is even-H-free where $c(H)$ is a constant dependent on H.

Let $g(n, H)$ denote the minimum number of colors in an edge-coloring of K_n that is even-H-free. Recent progress has been made on characterizing $g(n, H)$ for various graphs H. In [CH] it is proved that $g(n, K_4) = n^{o(1)}$. In [BHZ23] and independently in [GXZ] it is shown that $g(n, K_5) = n^{o(1)}$. In [Yi] it is proved that $g(n, K_8) = n^{o(1)}$. In [Ver24] it is proved that $g(n, H)$ is $n^{\Omega(1)}$ for graphs H with an even decomposition, where an even decomposition of a graph H is defined as a sequence of sets $V(H) = V_0 \supset V_1 \supset \cdots \supset V_k = \emptyset$ such that for all $i \in [k]$, the set $V_{i-1} \setminus V_i$ is independent and the number of edges in H between $V_{i-1} \setminus V_i$ and V_i is even. Other graph coloring problems, such as the one described in [Kru20], have direct implications for the study of $g(n, H)$ and thus for that of the properties of linear graph-codes.

1.1 Our Contribution

The present paper analyzes graph codes by studying the behaviour of $g(n, H)$, thus making partial progress towards the problems proposed in [Alo23b]. Our main new result is the identification of a family of graphs H satisfying $g(n, H) = \Theta(n^2)$. This is described in the following result.

Theorem 1.1. *Let $e(S)$ denote the number of edges of a star S and let $(S_1, S_2, \cdots, S_k)$ denote the star forest whose connected components are the stars $S_1, \ldots, S_k$ where $e(S_1) \geq e(S_2) \geq \cdots \geq e(S_k)$. Suppose $H = (S_1, S_2, \cdots, S_k)$ where $e(S_i)$ is even for all $i \in 1, 2, \cdots, k$. Then, we have the following:*

- *If $e(S_1) > e(S_2) + e(S_3) + \cdots + e(S_k)$, then $g(n, H) = \Theta(n)$*
- *If $e(S_1) \leq e(S_2) + e(S_3) + \cdots + e(S_k)$, then $g(n, H) = \Theta(n^2)$*

We additionally establish some general results about the function $g(n, H)$, but Theorem 1.1 describes the only class of graphs H for which we can show that $g(n, H) = \Theta(n^2)$. It is possible that there are other such graphs, one possible example could be forests of long even paths, but this is still an open problem.

2 Basic Properties of even-H-free colorings

For convenience, we restate the relevant definitions. We say that an edge coloring $\mathcal{C}$ of the complete graph K_n is *even-H-free* if there is no even copy of H in K_n, that is, every copy of H has at least one color appearing an odd number of times. Let $g(n, H)$ denote the least number of colors in an even-H-free coloring of K_n.

A related problem is that of determining the asymptotics of $f(K_n, H, q)$, which is the minimum number of colors needed to edge-color K_n so that every copy of H is colored with at least q colors. [Kru20] obtained the following result that will be useful for characterizing the behavior of $g(n, H)$:

Lemma 2.1. *([Kru20]) Let H be a graph with v vertices and e edges, then $f(K_n, H, q) = O\left(n^{\frac{v-2}{e-q+1}}\right)$.*

Combining this result with some basic combinatorial arguments we can derive several general properties of the function $g(n, H)$ as described in Lemma 2.2 below.

Lemma 2.2. *Let $g(n, H), n, H$ be as defined above. Then, we have the following:*

1. *If H is connected, then $g(n, H)$ is sub-quadratic (in n).*
2. *If H has exactly two vertices of odd degree, then $g(n, H) < 2n$.*
3. *Let $H = (H_1, H_2, \ldots, H_k)$ denote a graph with connected components H_i, $1 \leq i \leq k$. If any connected component H_i has an odd number of edges, then $g(n, H) \leq n - 1$.*

4. *Suppose $H = (H_1, H_2, \ldots, H_k)$ where each H_i is a connected component. If any H_i satisfies $e_i > 2v_i - 4$, where v_i and e_i denote the numbers of vertices and edges of H_i, respectively, then $g(n, H)$ is sub-quadratic.*

Proof. *(1)* If every copy of H has at least $q = \lfloor e/2 \rfloor + 1$ colors, then every copy of H has at least one color class of size 1. Additionally, because H is connected, $e \geq v - 1$. Plugging this result into Lemma 2.1 and using the fact that $\frac{v-2}{e-q+1} \leq \frac{e-1}{e-q+1} = \frac{e-1}{\lceil e/2 \rceil} < 2$ yields the desired result.

(2) Assign each vertex in K_n a distinct binary vector of length $\lceil \log_2 n \rceil$ and color every edge by the sum of the vectors assigned to its two endpoints. This coloring has at most $2n$ colors and is even-H-free, since the sum of vectors representing the colors of all edges of any copy of H is exactly the sum of the two vectors assigned to the two vertices of odd degrees, which is nonzero. This yields the desired result.

(3) Consider the coloring of K_n where we assign edge (i, j) the color $\min(i, j)$. Now, examine a copy of H in K_n. Note that the edges of its component H_i are assigned different colors than those of the other components H_j. Since some H_i has an odd number of edges, at least one color must appear an odd number of times in this H_i, showing that $g(n, H) \leq n - 1$.

(4) Consider the coloring (call it $\mathcal{C}$) prescribed by Lemma 2.1 for the graph H_i with $q = \lfloor e_i/2 \rfloor + 1$ where H_i is the connected component with $e_i > 2v_i - 4$. Note that

$$\frac{v_i - 2}{e_i - q + 1} < \frac{\frac{e_i + 4}{2} - 2}{e_i - \left(\lfloor \frac{e_i}{2} \rfloor + 1\right) + 1} = \frac{\frac{e_i}{2}}{\lceil \frac{e_i}{2} \rceil} \leq 1.$$

Thus $\mathcal{C}$ uses a sub-linear number of colors. Now, consider the product coloring $\mathcal{C} \times \min(i, j)$. Note that every copy of H_i has at least one color that appears exactly once in the coloring $\mathcal{C}$, and this color will never appear again in its copy of H because the $\min(i, j)$ component is different in the edges of the other components H_j. Thus, $g(n, H)$ is sub-quadratic. $\square$

By Lemma 2.2, it is apparent that graphs with $g(n, H) = \Theta(n^2)$ must be disconnected and sparse. A natural class of graphs to consider is the class of forests. It is easy to establish the following:

Lemma 2.3. *Suppose that $H = (T_1, T_2, \ldots, T_k)$ is a forest where each T_i is a tree. If the total number of edges in H is even, then $g(n, H) \geq \Omega(n)$.*

Proof. Consider a valid even-H-free coloring of K_n by C colors. Note that there cannot be a monochromatic H in this coloring. Since the Turán number of a forest H satisfies $ex(n, H) \leq v(H)n$, no color class can appear more than $v(H)n$ times. Thus, we must have $C \geq \frac{\binom{n}{2}}{v(H)n} = \Omega(n)$. Hence, $g(n, H) \geq \Omega(n)$. $\square$

It immediately follows that if a forest H has an even number of edges and at least one tree H_i has an odd number of edges, then $g(n, H) = \Theta(n)$. Therefore, the only forests (and in general graphs H) for which $g(n, H)$ may be quadratic in n must be sparse disconnected graphs in which every component has an even number of edges.

3 Proof of Theorem 1.1

Proof of Theorem 1.1 (first part). Observe that for any graph H (not necessarily a forest), if H has some vertex v with $\deg(v) > e(H)/2$, then $g(n, H) \leq n$. Indeed, if we color K_n via a proper edge coloring (in which every two adjacent edges have distinct colors), then each copy of H must have a color that appears exactly once, since the number of distinct colors of edges incident with the copy of v in H exceeds $e(H)/2$.

In the first case, the vertex v at the center of the star S_1 satisfies $\deg(v) > e(H)/2$. Thus, combining the lower bound from Lemma 2.3 with the upper bound above yields that in this case $g(n, H) = \Theta(n)$. □

In order to prove the second part, we need the following result, which is a variant of the dependent random choice lemma as described in [AS16]:

Lemma 3.1. *Consider a bipartite graph $G = (A \cup B, E)$ where the average degree of the vertices in A is at least d. Then, for any real parameter $p \in (0, 1)$, there exists a set K of size at least $p|A|\frac{d}{|B|} - \binom{|A|}{2}\frac{m}{|B|}p^2$ such that every two vertices $v_1, v_2 \in K$ have at least m common neighbors.*

Proof. Uniformly choose a random vertex $b \in B$, and for each vertex $a \in A$, randomly and independently, set $y_a = 1$ with probability p and 0 otherwise. Consider the set X, where $X = \{v \in A \mid (v, b) \in E, y_v = 1\}$ and let Y denote the set of (unordered) pairs of vertices in X which have less than m common neighbors.

For each vertex $a \in A$, the probability of b being its neighbor is $\frac{\deg(a)}{|B|}$ and the probability that $y_a = 1$ is p. By linearity of expectation, $\mathbb{E}|X| \geq p|A|\frac{d}{|B|}$. We next bound $\mathbb{E}[|Y|]$.

To do this, first define the fully deterministic set

$$S = \{(v_1, v_2) : (v_1, v_2) \in A, |N(v_1) \cap N(v_2)| < m\},$$

where each pair is unordered and composed of distinct vertices. Let $P(X)$ denote the (random) set of unordered pairs of vertices in X. Observe that by definition $|Y| = |S \cap P(X)|$. Thus, we have:

$$\mathbb{E}|Y| = \mathbb{E}|S \cap P(X)| = \sum_{s \in S} \mathbb{E}[\mathbb{1}[s \in P(X)]]$$

where $\mathbb{1}[\cdot]$ is the indicator function of the given event. Observe that for $(v_1, v_2) \in S$

$$\begin{aligned} \mathbf{Pr}\{(v_1, v_2) \in P(X)\} &= \mathbf{Pr}\{v_1, v_2 \in N(b), y_{v_1} = 1, y_{v_2} = 1\} \\ &= \mathbf{Pr}\{b \in N(v_1) \cap N(v_2), y_{v_1} = 1, y_{v_2} = 1\} \leq \frac{m}{|B|}p^2 \end{aligned}$$

where the last inequality follows from the definition of S and the random scheme defined in the first paragraph. This yields that $\mathbb{E}|Y| \leq |S|\frac{m}{|B|}p^2 \leq \binom{|A|}{2}\frac{m}{|B|}p^2$. Therefore

$$\mathbb{E}|X| - \mathbb{E}|Y| \geq p|A|\frac{d}{|B|} - \binom{|A|}{2}\frac{m}{|B|}p^2.$$

Fix a random choice for which

$$|X| - |Y| \geq p|A|\frac{d}{|B|} - \binom{|A|}{2}\frac{m}{|B|}p^2.$$

Finally, construct the set K by deleting one vertex from every pair in $Y \subset X$, achieving the desired result. $\square$

Proof of Theorem 1.1 (second part). Trivially, $g(n, H) \leq \binom{n}{2}$, as we can just assign every edge of K_n a unique color. It thus suffices to show that $g(n, H) \geq \Omega(n^2)$.

For notational convenience, let $e_1 \geq e_2 \geq \cdots \geq e_k$ denote the sizes of the trees in the star forest H, so $e_i = e(S_i)$. Define $e = \sum_{i=1}^{k} e_i$ and $v = \sum_{i=1}^{k}(e_i + 1)$ to be the number of edges and vertices of H respectively. Now, assume (for the sake of contradiction), that there exists an even-H-free coloring $\mathcal{C}$ that utilizes less than ϵn^2 colors (where ϵ is some constant that we will set later).

Within the coloring $\mathcal{C}$, we call an edge *"problematic at v"* if there exists another edge of the same color incident to it at v. Observe that if there exists more than $\frac{3}{2}(e_1 - 1)$ problematic edges at v, then there exists an even star S_1 centered at v, that is, a copy of S_1 centered at v in which every color appears an even number of times. Consider the following process: For every vertex v with at least $\frac{3}{2}(e_1 - 1)$ edges that are *problematic at v*, take the vertex v and the e_1 vertices neighboring it that form an even copy of S_1 and remove these $e_1 + 1$ vertices. Note that we can only conduct this process at most k times as otherwise, there will be an even H. Thus, during this process, we have removed less than $ke_1 < n/4$ vertices in total. Thus, we are left with a (complete) graph G' that has at more than $3n/4$ vertices remaining, where each vertex has less than $\frac{3}{2}e_1$ problematic edges incident with it.

Denote $|V(G')| = n' > 3n/4$. Construct a graph G'' by deleting all problematic edges from G. Note that G'' is properly edge-colored and that the average degree of G'' is at least $n' - 3e_1 > n/2$. Now, consider the auxiliary bipartite graph $B = (L \cup R, E)$ where L is a set of $n/2$ vertices in $V(G'')$ and $R = \mathcal{C}$ and the edges are all pairs (v, c) where vertex v is incident to an edge of color c in G''.

We now aim to show that there exists a set of vertices in G'' such that every pair of vertices has at least m common colors (where m will be chosen later). As this is equivalent to showing that there exists a set of vertices in $L \subseteq B$ such that every pair of vertices has m common neighbors, we can apply Lemma 3.1 with parameter $p = \frac{1}{2m}$ to get that there exists such a set of size at least

$$\frac{1}{2m} \cdot \frac{n}{2} \cdot \frac{n/2}{\epsilon n^2} - \frac{n^2}{8} \cdot \frac{m}{\epsilon n^2} \cdot \frac{1}{4m^2} = \frac{1}{8m\epsilon} - \frac{1}{32m\epsilon} > \frac{1}{16m\epsilon}$$

Therefore, we have a set S of size $\frac{1}{16m\epsilon}$ such that every pair of them have at least m common neighbors. Setting $m = 2(e + k)$ and $\epsilon = \frac{1}{32(e+k)k}$ yields that S has size k. Let $S = \{v_1, v_2, \cdots, v_k\}$ be the centers of the k stars of H and delete all $\binom{k}{2}$ edges between vertices of S. Observe that every two distinct vertices v_i and v_j vertices in S has at least $2e$ colors in common amongst the edges that connect them to other edges. All that remains is to assign these edges correctly to form an even copy of H.

Construct a sequence $y_1 y_2 \cdots y_e$ of length $e = \sum_{i=1}^{k} e_i$, where each element in the sequence is assigned a number between 1 and k as follows. The first e_1 elements of the sequence are 1, the next e_2 elements to be 2, and so on until the last e_k elements are all k. Now, we will assign edges in $e/2$ steps, where in step i we will add edges of the same color to v_{y_i} and $v_{y_{i+e/2}}$ and then delete the vertices corresponding to the two leaves that we have added. Note that $y_i \neq y_{i+e/2}$ since $e_i \leq \frac{1}{2}e$ by the inequality constraint assumed. In each step, we are deleting at most 2 of the colors incident with each v_i, which means that throughout the process, we are deleting at most $2 \cdot e/2 = e$ edges from each v_i. It follows that at each step, a given pair (v_i, v_j) will lose at most 4 common neighbors. Since each pair has at least $m \geq 2e$ common colors without the edges connecting the v_i's themselves, we will always have a common color when needed, guaranteeing that the procedure terminates. As the procedure assigned vertex v_i exactly e_i edges, and all color classes are even (as they were assigned to centers in pairs), there exists an even H, which contradicts $\mathcal{C}$ being even-H-Ramsey coloring. $\square$

4 Influence of Star forest parameters

Consider H prescribed by the second part of Theorem 1.1, that is, an even forest satisfying $e(S_1) \leq e(S_2) + e(S_3) + \cdots + e(S_k)$. Our proof in the prior section showed that any coloring with less than $\frac{1}{32k(e+k)}n^2$ colors cannot be even-H-Ramsey. In this section, we analyze special cases of star forests where we can more precisely characterize the effects of the parameters e and k on $g(n, H)$.

Theorem 4.1. *Suppose $H = (S_1, S_2)$ where S_1 and S_2 are both disjoint even stars. Then, we have the following:*

- *If $e(S_1) > e(S_2)$, then $g(n, H) = \Theta(n)$*
- *If $e(S_1) = e(S_2)$, then $g(n, H) = \Theta(\frac{n^2}{e})$*

To prove Theorem 4.1, we require the following classical result of Wilson about graph decomposition.

Lemma 4.2. *([Wil76]) For every finite simple graph T there exists a threshold $n_0(T)$ such that if $n > n_0(T)$ and the following two conditions hold then the edge set of the complete graph K_n can be partitioned into subgraphs each of which is isomorphic to T. The two conditions are:*

1. *$\binom{n}{2}$ is divisible by $|E(T)|$*
2. *$n - 1$ is divisible by the greatest common divisor of the degrees of vertices in T*

Proof of Theorem 4.1. The first part directly follows from Theorem 1.1. Additionally, for the second case, substituting $k = 2$ into the lower bound $\frac{n^2}{32k(e+k)}$ proved above implies that $g(n, H)$ is at least $\Omega(\frac{n^2}{e})$. All that remains is to show that $g(n, H)$ is at most $\Omega(\frac{n^2}{e})$.

Denote $e(S_1) = e_1 = e/2$. Consider n that meets the prescribed conditions for Lemma 4.2 with $T = K_{2e_1+1}$, that is $n > n_0(K_{2e_1+1})$, $\binom{2e_1+1}{2} \mid \binom{n}{2}$, and $2e_1 \mid n-1$. Now, partition the edge set of K_n into $\binom{n}{2}/\binom{2e_1+1}{2}$ copies of K_{2e_1+1} and properly edge color each K_{2e_1+1} with $2e_1 + 1$ unique colors so that no two distinct copies of K_{2e_1+1} in the decomposition have any colors in common. Denote this coloring by $\mathcal{C}$

Suppose that there is an even copy of H in which u and v are the centers of the two stars. The edge (u, v) belongs to a unique copy $\mathcal{K}$ of K_{2e_1+1} in the decomposition. Therefore, the only possibility for an edge incident with u and an edge incident with v that have the same color is that both these edges lie inside $\mathcal{K}$. But, this requires all of H to lie inside $\mathcal{K}$, which is impossible as $|V(H)| = 2e_1 + 2 > 2e_1 + 1 = V(\mathcal{K})$. It thus follows that $\mathcal{C}$ is even-H-free. Note also that $\mathcal{C}$ can be extended to all $n > n_0(K_{2e_1+1})$. This is because we first choose the smallest $n' > n$ that meets the two conditions for Lemma 4.2, color $K_{n'}$ with $\mathcal{C}$, and then delete $n' - n$ vertices to attain an even-H-free coloring for K_n.

As the number of colors in $\mathcal{C}$ is $(2e_1 + 1) \cdot \binom{n}{2}/\binom{2e_1+1}{2}$, it follows that under the divisibility conditions $g(n, H)$ is at most $\frac{n^2}{2e_1}$, implying the desired result. □

Remark 4.3. We also observe that when H is composed of k copies of S_2, then for all k, $g(n, H) \geq n^2/10$. This is easy to see by iteratively finding even copies of (S_2, S_2) and (S_2, S_2, S_2) as prescribed by Theorem 1.1 and deleting them until we have found even H.

A similar argument follows for general forest of stars $H = (S_1, S_2, \cdots S_k)$ that can be partitioned into subsets $P_1, P_2, \cdots P_l$ of H where each subset $P_i = (S_{i1}, S_{i2}, \cdots, S_{im})$ satisfies $e(S_{i1}) \leq e(S_{i2}) + e(S_{i3}) + \cdots + e(S_{im})$.

5 Conclusion

We tried to characterize the graphs H for which even-H-free colorings require many colors. In Section 2, we looked at some more general cases and identified that graphs with high $g(n, H)$ must be sparse and disconnected.

Theorem 1.1 fully characterizes the behavior of the function $g(n, H)$ for forests of even stars. In particular, it shows that when forests of even stars satisfy an appropriate size constraint, then $g(n, H) = \Theta(n^2)$.

This paper makes progress in the study of one of the open questions suggested in [Alo23b]. However, there are still unanswered questions pertaining to even-H-free colorings and their broader applications. The following specific problems and conjectures are interesting.

Conjecture 5.1. The only class of graphs for which $g(n, H) = \Theta(n^2)$ are those described in Theorem 1.1.

Problem 5.2. What is the asymptotic behavior of $g(n, H)$ where H is a collection of disjoint even paths?

Conjecture 5.1 and *Problem 5.2* are closely related as it is possible that $g(n, H)$ is large for a collection of disjoint even paths (see [Kru20]). It could even be that $g(n, H) = \Theta(n^2)$ in this case, but at the moment this remains open.

Although this paper focuses on the study of graphs H for which $g(n, H)$ is large, the investigation of the cases for which $g(n, H)$ is small is also interesting. The known results about even decomposition of random graphs proved in [Ver24], [JY] eliminate most graphs from having $g(n, H) = n^{o(1)}$. On the other hand it is known that K_4, K_5, and K_8 have $g(n, H) = n^{o(1)}$ (see [CH], [BHZ23], [GXZ], [Yi]). It is likely to expect that this holds for every fixed complete graph (with an even number of edges).

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